Critical Behaviour in the Relaminarisation of Localised Turbulence in Pipe Flow

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The statistics of the relaminarisation of localised turbulence in a pipe are examined by direct numerical simulation. As in recent experimental data (Peixinho & Mullin *Phys. Rev. Lett.* **96**, 094501, 2006), the half life for the decaying turbulence is consistent with the scaling $(Re_c - Re)^{-1}$, indicating a boundary crisis of the localised turbulent state familiar in low-dimensional dynamical systems. The crisis Reynolds number, is estimated as $Re_c = 1870$, a value within 7% of the experimental value 1750. We argue that the frequently-asked question of which Re and initial disturbance are needed to trigger sustained turbulence in a pipe, is really two separate questions: the 'local phase space' question (local to the laminar state) of what threshold disturbance at a given Re is needed to initially trigger turbulence, followed by the 'global phase space' question of whether Re exceeds Re_c at which point the turbulent state becomes an attractor.

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Understanding the behaviour of fluid flow through a circular straight pipe remains one of the outstanding problems of classical physics and has continued to intrigue the physics community for more than 160 years [1],[2],[3],[4]. Although all evidence indicates that the laminar parabolic flow is linearly stable, the flow can become turbulent even at modest flow rates. The exact transition point depends not only on the flow rate (measured by the Reynolds number $Re = UD/\nu$, where U is the axial flow speed, D is the pipe diameter and ν is the fluid's kinematic viscosity) but also sensitively on the shape and amplitude of the disturbance(s) present [5], [6], [7], [8]. When it occurs, transition is abrupt with the flow immediately becoming temporally and spatially complex. Given that most industrial pipe flows are turbulent and hence more costly to power than if laminar, a central issue is to understand the conditions which trigger sustained turbulence. The problem is, however, severely complicated by the fact that the threshold appears very sensitive to the exact form of the disturbance *and* long turbulent transients can exist close to the threshold. Of particular interest is the low-Re situation where the transition typically leads to a clearly localised turbulent structure called a 'puff' within the laminar flow [3], [9]. A puff has a typical length of about $20D$ along the pipe (see Fig. 1) and, despite appearing established, can relaminarise without warning after travelling many hundreds of pipe diameters downstream.

There have been a number of contributions to this problem but so far no consensus on the minimum Reynolds number, Re_c, above which turbulence is sustained. Experimental studies have focussed on plotting transition-threshold curves in disturbance amplitude-Re space for specific forms of applied perturbation. One well-studied perturbation having six-fold rotational symmetry gave rise to a threshold amplitude which scaled like Re^{-1} above $Re = 2000$ [6] but diverged at $Re_c \approx 1800$ [7], i.e. below this value no sustained turbulence could be excited however hard the flow was disturbed. Subsequent experiments [8] studying the statistics of relaminarisations of puffs as Re is reduced have lowered this threshold value to $Re_c = 1750 \pm 10$, close to a previous estimate of 1760 [5] but not to others of 1876 [10] and ≈ 2000 [9]. The only complementary numerical work performed so far has been in a short periodic pipe of $5D$ length [11] where it was demonstrated that the pipe-long turbulent state displays the transient characteristics of a chaotic repellor until $Re_c = 2250$ above which it becomes a chaotic attractor. Recent experiments using a very long pipe [12] in which the statistics on long transients are available, however, suggests that there is no critical behaviour. Rather than the turbulent half life scaling like $\tau \sim (Re_c - Re)^{-1}$ [8],[11], it is found to increase exponentially instead. Interestingly, re-interpretation of the $5D$-pipe data seems to corroborate this alternative exponential lifetime behaviour even though the pipe is too short to capture a turbulent puff.

In this Letter, we consider a much longer pipe of length $16\pi D$ ($\approx 50D$) in which turbulent puffs can be represented faithfully using direct numerical simulation [13] and examine the statistics of how they relaminarise. We find an exponential distribution of lifetimes and the critical scaling law $\tau \sim (Re_c - Re)^{-1}$, with a constant of proportionality and an estimate of $Re_c = 1870$ both in good agreement with experimental data [8]. Surprisingly, given its long history, this represents the first time that a *quantitative* connection between theory and experiment has been established in the pipe flow problem.

The Navier–Stokes equations for an incompressible Newtonian fluid,

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p = \nu \nabla^2 \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

in a straight pipe with circular cross-section and for constant mass-flux, were solved numerically in cylindrical coordinates (r, θ, z) using a mixed pseudospectral-finite difference formulation [27]. The code was found to accurately reproduce linear stability results for Hagen-Poiseuille flow, instabilities of nonlinear travelling wave

FIG. 1: Numerical ‘puff’ at $Re = 1900$. (r, z)-section of $(\boldsymbol{\nabla} \times \boldsymbol{u})_z$. Only $20D$ shown of $50D$ computational domain.

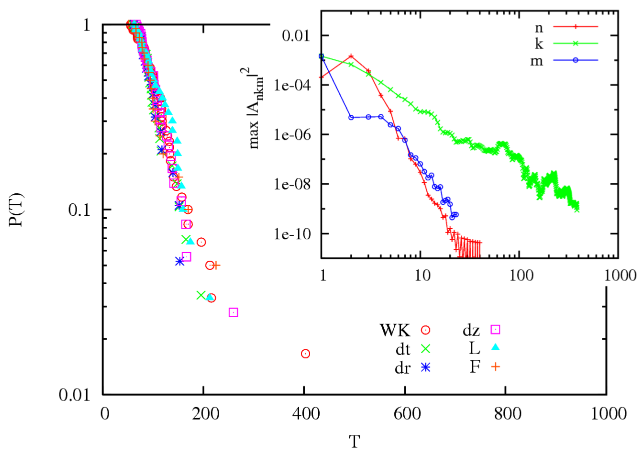

FIG. 2: Probability of relaminarisation after time T at $Re = 1740$ is the same for increased resolutions (‘dt’ data with timestep halved, ‘dr’ data with 60 radial points, ‘dz’ data with axial resolution of ± 576), pipe length (‘L’ is $100D$ data), and different disturbance (‘F’ is data obtained with the puff generated by an initial period of body forcing - the data is shifted in the last case to account for a longer transient period). All data sets effectively overlay the default data ’WK’. Inset, numerical puff spectrum at $Re = 1900$, $A_n = \max_{km} |A_{nkm}|^2$, index n of Chebyshev transformed radial modes, k, m axial and azimuthal Fourier modes respectively and similarly for A_k and A_m.

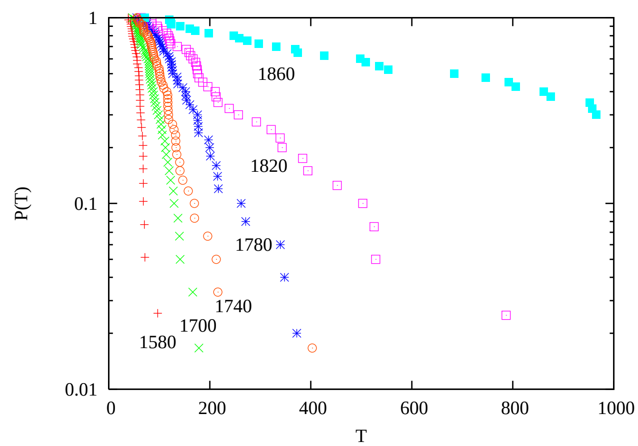

FIG. 3: The probability of turbulent lifetime $\geq T$, $P(T)$, for several Re in a periodic pipe of length $16\pi D$.

solutions and the statistical properties of turbulent pipe flow [14] (as well as being cross-validated with another code [15]). A resolution of 40 radial points was adopted with grid points concentrated at the boundary, Fourier modes were kept up to ± 24 in θ, and to ± 384 in z for a periodic pipe of length $L = 16\pi\, D$. This ensured spectral drop off of 6 orders of magnitude in the power of the coefficients when representing a puff velocity field at $Re = 1900$: see inset in Fig. 2. The timestep was dynamically controlled using information from a predictor-corrector method and was typically around $0.006\, D/U$. The initial conditions for the calculations were randomly-selected velocity snapshots taken from a long puff simulation performed at $Re = 1900$. A body forcing applied over $10D$ of the pipe and for a time $10\, D/U$ was used to generate an ‘equilibrium’ puff which remained stable in length and form for a time period of over $2000\, D/U$ (see Fig. 1). At a chosen $Re < 1900$ a series of at least 40 and up to 60 independent simulations were performed each initiated with a different puff snapshot to generate a data set of relaminarisation times. The signature of the relaminarisation was a clear and sudden transition to exponential decay of the energy. The criterion for relaminarisation was taken to be such that the energy of the axially-dependent modes was less than $5 \times 10^{-4}\, \rho U^2 D^3$, below which all solutions were well within the decaying regime. The range of measured Re_c discussed above indicates sensitivity to noise. Robustness of the relaminarisation statistics was verified by comparing the half-lives of data sets obtained by varying different computational parameters of the simulation (see Fig. 2). All modifications produce half-life values within the 95% confidence interval about the default half-life prediction.

Decay probabilities for a range of Reynolds numbers are shown in Fig. 3 over an observation window of $1000\, D/U$. The linear drop-off of the probability on the log-plot strongly suggests the exponential distribution, $P(T) \sim \exp(-T \ln 2/\tau)$, where $\tau = \tau(Re)$ is the half-life of a puff. The median of $(T - t_0)$ was used as an estimator for τ. Inspection of the data by varying the cut-off time t_0 revealed the effects of an initial transient period in the first few data points. This was minimised by selecting t_0 to exclude the first 5-10% of the data (determined by looking for the least sensitivity in the half-life prediction). The results plotted in Fig. 4 are consistent with the relation $\tau = \alpha(Re_c - Re)^{-1}$ where α is 2.4×10^{-4} compared to 2.8×10^{-4} obtained in [8] and there is a shift of 7% in Re_c up to 1870 in the numerical data. Also shown is the reinterpreted numerical data for the $5D$ pipe [11] and the recent half-life results from the long pipe experiments [12] which indicate that $1/\tau$ varies exponentially with Re rather than linearly. Although the data from [12] is for longer times, there is sufficient over-

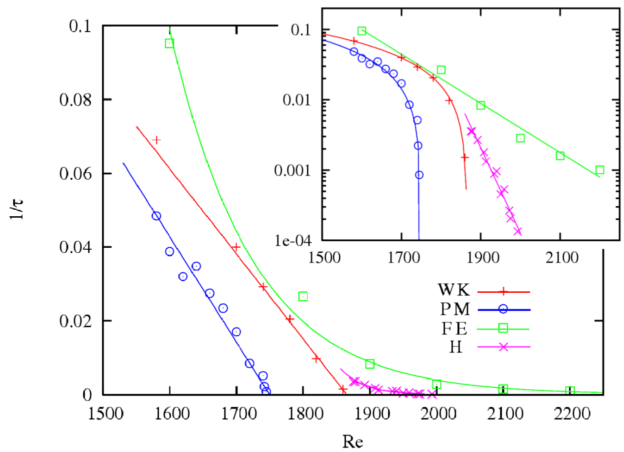

FIG. 4: The reciprocal of the puff half-life τ plotted against Re. Data plotted: 'WK'- $50D$ data (each data point is the result of 40-60 simulations); 'PM' - experimental data from [8]; 'FE' - reinterpreted $5D$ data [11]; 'H' - experimental data from [12]. Inset, log-plot of $1/\tau$ vs Re.

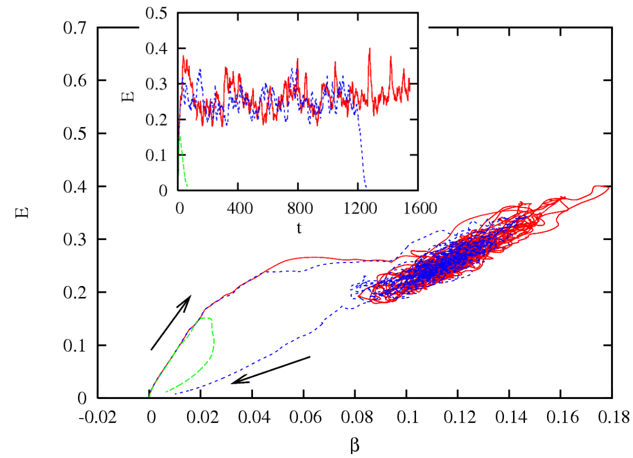

FIG. 5: Trace of perturbation energy versus additional pressure fraction required to maintain fixed mass flux, $1+\beta = <\partial_z p> / \mathrm{d}_z p_{lam}$ (the origin represents laminar flow), for the three cases of a sustained puff at $Re = 1900$ (solid), metastable puff at $Re = 1860$ with sudden relaminarisation (dotted) and the immediate decay of a perturbation (dashed). The inset shows that the energy trace for the metastable puff is similar to the sustained puff before it laminarises.

lap to suggest that the data from [8] and our results are not consistent with being the earlier linear-looking part of this exponential. Rather, the results indicate qualitatively different behaviour [28].

The exponential probability distribution $P(T)$ found here in Fig. 3 implies that puff relaminarisation is a *memoryless* process - the probability that the puff will decay in a given interval of time is proportional to the length of the period but independent of previous events. This feature has been found previously in turbulent relaminarisation experiments in pipe flow [8],[12],[16] as well as in plane Couette flow [17],[18] and numerical calculations using models of this together with other linearly stable shear flows [19],[20]. Faisst and Eckhardt [11] interpret this result as indicating that the transient turbulent state for $Re < Re_c$ represents a chaotic repellor in phase space. Our results indicate that this conclusion carries over to a localised turbulent puff in a long pipe. The building blocks for such a repellor are saddle points and families of these in the form of travelling waves with discrete rotational symmetries are now known to exist down to $Re = 1251$ [21], [22], [23]. Tentative experimental evidence for their relevance to puffs has already been found [24] and corroborating numerical evidence is now emerging [15]. The entanglement of all the stable and unstable manifolds associated with these saddles at some higher Re presumably gives rise to sufficiently complicated phase dynamics to appear as a turbulent puff in real space. That this phase space structure is initially 'leaky' ultimately allowing escape (relaminarisation) is perhaps unsurprising but what is less clear is how it suddenly becomes an attractor at Re_c. The clean scaling of the transient decay half life, $\tau \sim (Re_c - Re)^{-1}$ strongly suggests a boundary crisis [25] while the precise value of the critical exponent hints at a simple dynamical systems explanation. One, of course, cannot rule out the possibility that the region never becomes an attractor with the exit probability becoming extremely small but staying finite as Re increases [12]. Or, in fact, that there are a number of 'leaks' which one by one seal up giving a half-life behaviour which varies over a number of discrete time scales. Also, at some point, the effect of noise must surely become significant over long times. However, the fact that the numerical simulations and the experimental results [8] are quantitatively consistent despite being subject to different types of errors/disturbances indicates that noise is not important over timescales of $O(1000\,D/U)$ for the levels maintained here and in the experiments.

The simulations confirm that the puff characteristics are continuous as Re crosses Re_c and that a puff corresponds to a part of phase space disjoint from the laminar state (see Fig. 5 and inset). This observation naturally divides the usual question as to how to trigger sustained turbulence in pipe flow into two separate issues. Firstly, what disturbance at a given Re is needed to trigger turbulence initially — i.e. what initial conditions will cause the flow to leave the neighbourhood of the laminar state to reach the puff region of phase space. And secondly, what Re is needed so that, for a flow already in the turbulent region, the flow never leaves — i.e. the puff has become an attractor. The implications of this realisation are that experimental curves in [5] and [7] showing a threshold curve on a disturbance amplitude-Re plot must, in fact, be two curves as shown in Fig. 6. Figure 5 shows how initially a threshold amplitude of disturbance

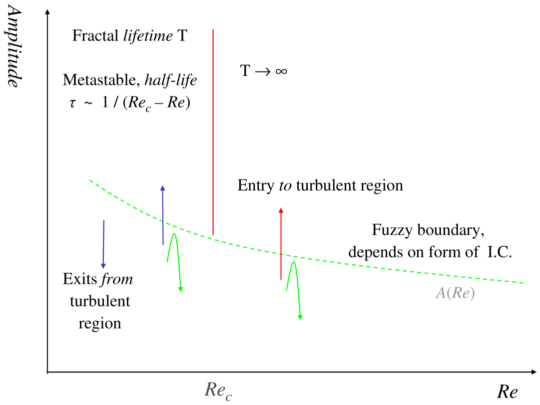

FIG. 6: Sketch of the two (independent) thresholds associated with transition: one is amplitude-dependent (and highly form-dependent) indicating when a turbulent episode is triggered, and the other is a global Re-dependent threshold indicating when the turbulence will be sustained.

is required to push the solution away from the laminar state and into the turbulent region. Once here, the exit from the metastable state is sudden and unrelated to the entry as relaminarisation is a memoryless process.

To summarise, numerical simulations described in this Letter have clarified the existence of two independent thresholds for sustained turbulence. Results probing the relaminarisation threshold closely match a recent experimental investigation [8]. For timescales extending these experiments — $t \leq 1000\,D/U$ — we confirm the presence of an exponential distribution for the probability of puff relaminarisation and corroborate critical-type behaviour in which the puff half-life diverges as $(Re_c - Re)^{-1}$. Good quantitative agreement between the experimentally and theoretically-estimated value of Re_c (less than 7% difference) is a rare triumph in this famous canonical problem.

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[27] Incompressibility was satisfied automatically by adopting a toroidal-poloidal potential formulation [26], further reformulated into five simple second order equations in r. The numerical discretization was via a non-equispaced 9-point finite difference stencil in r and by Fourier modes in θ and z. At the pipe wall boundary conditions coupling the potentials were solved to numerical precision using an influence-matrix method, and axial symmetry properties imposed by the geometry on each Fourier mode were enforced implicitly in the finite difference weights.
[28] In [12], the flow is disturbed by a jet of injected fluid much as in earlier experiments [7] where a six-jet disturbance was used. This latter study found that results were sensitive to the exact flux fraction of the laminar flow injected, with a (large) value of 0.1 giving $Re_c = 1710 \pm 10$ whereas a (small) disturbance of 0.01 gave $Re_c = 1830 \pm 10$: [12] quote injected flux rates of ≈ 0.07.